

Two One-Parameter Special Geometries

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Abstract

The special geometries of two recently discovered Calabi-Yau threefolds with $h^{11} = 1$ are analyzed in detail. These correspond to the ‘minimal three-generation’ manifolds with $h^{21} = 4$ and the ‘24-cell’ threefolds with $h^{21} = 1$. It turns out that the one-dimensional complex structure moduli spaces for these manifolds are both very similar and surprisingly complicated. Both have 6 hyperconifold points and, in addition, there are singularities of the Picard-Fuchs equation where the threefold is smooth but the Yukawa coupling vanishes. Their fundamental periods are the generating functions of lattice walks, and we use this fact to explain why the singularities are all at real values of the complex structure.

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1. Introduction

Detailed descriptions of special geometries are known only in relatively few cases. Explicit examples have been limited to dimension ≤ 3 , with the most explicit examples of 1 and 2 dimensions. In the Kreuzer-Skarke list [1, 2] there are 5 manifolds with $h^{2,1} = 1$. All of these share features with the parameter space of the mirror quintic in that the parameter space is a \mathbb{P}^1 with three singular points: a large complex structure limit, with maximal unipotent monodromy, a Gepner point, with \mathbb{Z}_n monodromy, for some n , and a conifold point. Doran and Morgan [3] have classified the 14 possible one-parameter special geometries with three singular points of the these types. Rodriguez-Villegas arrived at this same list [4] by means of number-theoretic considerations. We now know [5–7] Calabi-Yau manifolds corresponding to each of these.

The two manifolds whose parameter spaces we examine here are a manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$, which is realised as the quotient of a hypersurface in $dP_6 \times dP_6$ by a freely acting group G of order 12. There are, in reality, two variants of the is first ‘manifold’ since there are two choices for G , which is either \mathbb{Z}_{12} or the nonabelian group Dyc_3 . This space can be realised as the CICY

$$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}^{(1,4)} / G$$

or as a reflexive polyhedron, corresponding to the fact that the covering space is a hypersurface in the toric variety $dP_6 \times dP_6$. A detailed description can be found in [8, 9].

The other manifold has $(h^{1,1}, h^{2,1}) = (1, 1)$ and occupies a very special position at the very tip of the distribution of Hodge numbers. This manifold was found by one of the present authors (VB) by applying the methods that realise the symmetry of the covering space of the (1,4)-manifold as an operation on the toric polyhedron. More precisely, it was known that one of the polytopes of the Kreuzer-Skarke list is the 24-cell, a regular self-dual polytope that has 24 vertices and 24 three-faces. This polytope has a large symmetry group. There are three subgroups, G , with the property that G acts freely and transitively on the vertices. One checks that this action on the vertices leads to a free action on the manifold. The most general polynomial invariant under G is thus of the form $P - \varphi Q$, where P is the sum, with unit coefficients, of the monomials corresponding to the vertices of the polyhedron and Q is the product of all the coordinates, and is the monomial corresponding to the interior point of the polytope. Since there is just one complex structure parameter, $h^{2,1} = 1$, for the quotient, and since $\chi = 0$ for the quotient, we also have $h^{1,1} = 0$. Full details of this construction may be found in [10].

The reason for presenting the special geometries of these two manifolds together is that they are rather similar, even though there is no, currently known, direct relation between the corresponding Calabi-Yau geometries. In each case there is a point of maximally unipotent monodromy, corresponding to the large complex structure limit, there is no Gepner point, so

no point with finite monodromy, and both of the special geometries have 6 (hyper-)conifold points. There are also singular points of the Picard-Fuchs equations for which the monodromy is trivial. There are two such points for the (1,4)-manifold and one for the (1,1)-manifold. These are nonsingular points of the parameter space where the Yukawa couplings vanish.

Another matter that we find to be of interest has to do with the relation between two bases for the periods. For each of our two spaces we have a fourth order differential operator \mathcal{L} . We denote by φ , in each case, a coordinate for the special geometry, and choose φ such that the large complex structure limit corresponds to $\varphi = 0$. The indices for \mathcal{L} , at this point, all vanish, so a basis of solutions for a neighbourhood of $\varphi = 0$ has the form

$$\begin{aligned}\varpi_0 &= h_0 \\ \varpi_1 &= \frac{1}{2\pi i} h_0 \log \varphi + h_1 \\ \varpi_2 &= \frac{1}{(2\pi i)^2} h_0 \log^2 \varphi + \frac{2}{2\pi i} h_1 \log \varphi + h_2 \\ \varpi_3 &= \frac{1}{(2\pi i)^3} h_0 \log^3 \varphi + \frac{3}{(2\pi i)^2} h_1 \log^2 \varphi + \frac{3}{2\pi i} h_2 \log \varphi + h_3 ,\end{aligned}\tag{1.1}$$

where the h_j are power series in φ . As it stands, the basis is not unique¹, but it becomes so if we require the boundary conditions $h_0(0) = 1$ and $h_j(0) = 0$. With these conditions, we refer to this basis as the Frobenius basis. We also assemble these periods into a vector

$$\varpi = \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix} .\tag{1.2}$$

We may also form another vector of periods Π corresponding to a symplectic basis of $H^3(M, \mathbb{Z})$. Interest then attaches to the constant matrix ρ that relates these vectors.

$$\Pi = \rho \varpi .$$

Now we can explain an observation that could have been made already with respect to the parameter space of the mirror quintic, but it arises here rather more dramatically because of the need for numerical computation. Consider the effect of continuing the vector ϖ around a loop that starts near $\varphi = 0$ and encloses one of the conifold points. With this loop we associate a monodromy matrix m

$$\varpi \longrightarrow m \varpi$$

Since we do not know the periods in closed form, for the manifolds we consider here, we must use numerical methods to perform the analytic continuation. We can, for example, integrate the equations $\mathcal{L}\varpi = 0$ numerically around a contour, that encloses the conifold

¹Since we may add multiples of ϖ_0 to ϖ_1 , multiples of ϖ_0 and ϖ_1 to ϖ_2 , and, finally, multiples of ϖ_0 , ϖ_1 and ϖ_2 to ϖ_3 .

point. We obtain in this way a numerical matrix m . For the (1,1)-manifold the matrix we obtain in this way is immediately seen to be a matrix of rational numbers of low height. For the (4,1)-manifold, by contrast, this is not so. Some of the elements of m are clearly rational numbers, but others are not. Since we know that the special geometry prepotential contains a term, with the universal form, $\chi\zeta(3)/(2\pi i)^3$, where χ is the Euler number of M , we hypothesize that the irrational elements of m are (linear combinations of) powers of this number multiplied by rational numbers of low height. In other words the elements of m take values in the field

$$\mathbb{K} = \mathbb{Q} \left[\frac{\chi\zeta(3)}{(2\pi i)^3} \right] ,$$

of rational numbers extended by $\chi\zeta(3)/(2\pi i)^3$. The results of numerical computation are that this hypothesis is correct. The reason for including χ in this statement is that if $\chi=0$, as it is for the (1,1)-manifold, then the assertion is that the elements of m take values in \mathbb{Q} .

Now if we transport the period vector Π around the same loop, we have

$$\Pi \longrightarrow M \Pi ,$$

with M a matrix of integers. Since we also have

$$m = \rho^{-1} M \rho$$

it must be the case that ρ takes values in \mathbb{K} .

In the following two sections we study the special geometry of the (4,1) and (1,1) manifolds. We find the respective Picard Fuchs equations by first finding the series expansion of the fundamental period and then seeking the fourth order differential equation that it satisfies. Once we have the Picard Fuchs operator \mathcal{L} we are able to calculate the various monodromy matrices as indicated above. The relation between the Frobenius basis and integral basis is determined by choosing the matrix ρ so that the monodromy matrices $\rho m \rho^{-1}$ are integral.

The calculation starts with the calculation of a sufficient number of coefficients a_n that appear in the series expansion of the fundamental period.

$$\varpi_0(\varphi) = \sum_{n=0}^{\infty} a_n \varphi^n .$$

Since the mirror manifolds form a 1-parameter family we expect ϖ_0 to satisfy a fourth-order linear differential equation with regular singularities. We therefore seek a differential operator, with polynomial coefficients,

$$\mathcal{L} = R_4(\varphi) \vartheta^4 + R_3(\varphi) \vartheta^3 + R_2(\varphi) \vartheta^2 + R_1(\varphi) \vartheta + R_0(\varphi) , \quad \text{where} \quad \vartheta = \varphi \frac{d}{d\varphi} , \quad (1.3)$$

that annihilates ϖ_0 . To this end we choose a maximal degree, k_{\max} , for the polynomials

$$R_j(\varphi) = \sum_{k=0}^{k_{\max}} r_{jk} \varphi^k$$

and solve the linear equations, that result from requiring that $\mathcal{L}\varpi_0$ should vanish, for the coefficients r_{jk} . If k_{\max} is chosen too small then the only solution is that the R_j vanish identically. We increase k_{\max} until we find a nontrivial solution. For the (4,1)-manifold we find $k_{\max} = 8$ and for the (1,1)-manifold we find $k_{\max} = 7$. In order to find these operators, and include some extra terms, as a check, we require the first 40 terms, say, in the expansion of ϖ_0 . A brute force calculation of this many terms of the series is difficult, so a more efficient method is required. For the cases we study we are able to calculate a sufficient number of coefficients by ad hoc methods. Once we have the Picard-Fuchs equation, however, we can compute many more coefficients a_n by using the differential equation to find a recurrence relation.

Let us employ an multi-index notation and denote monomials that appear in the toric (Laurent) polynomial that defines the manifold by $t^v = \prod_j t_j^{v_j}$, in terms of four variables t_j . The exponents v define a convex lattice polyhedron Δ , with the origin $v=0$ as an interior point. With these conventions, one parameter families of Calabi–Yau hypersurfaces in toric varieties, such as the two families we consider here, are described by the vanishing of polynomials of the form

$$1 - \varphi \sum_{v \in \Delta \setminus \{0\}} t^v . \quad (1.4)$$

The exponents that appear in the defining polynomial define also a semigroup

$$\mathcal{S} = \left\{ \sum_{v \in \Delta} n_v v \mid n_v \in \mathbb{N} \right\}$$

for which, by definition, all the coefficients n_v are nonnegative. For the Calabi-Yau hypersurfaces that we will consider in this paper, the universal cover is a hypersurface in a particularly symmetric toric variety. In fact, Δ will be the fundamental region of a lattice, that is, the translates of Δ tile \mathcal{M} . In that case we can also describe the vertices as the nearest neighbour lattice points of the origin and the semigroup \mathcal{S} coincides with the lattice \mathcal{M} .

Consider now the process of counting the number of lattice walks, for \mathcal{M} , from the origin to a lattice site u , in n steps. The answer is given by the coefficient of $t^u \varphi^n$ in the multivariate generating series

$$\sum_{n=0}^{\infty} \left(\varphi \sum_v t^v \right)^n = \frac{1}{1 - \varphi \sum_v t^v}$$

where the generating function is written in terms of four formal variables t_j , and the sum over v is taken over $v \in \Delta \setminus \{0\}$. For a toric hypersurface defined by a polynomial of the form (1.4) it can be shown that coefficients, a_n , in the series expansion of the fundamental period

$$\varpi_0(\varphi) = \sum_{n=0}^{\infty} a_n \varphi^n$$

are given by the coefficient of $\varphi^n t^0$ in the expansion of

$$\frac{1}{1 - \varphi \sum_v t^v} .$$

Thus a_n counts the number of lattice walks that return to the origin after n steps and ϖ_0 is the corresponding generating function.

We study the parameter space of the (1,4)-manifold in §2 and the (1,1)-manifold in §3. In §4 we return to the interpretation of ϖ_0 as the generating function of lattice walks that return to the origin after n steps.

The coefficients of the periods encode also interesting number-theoretic information about the underlying Calabi–Yau manifolds. A knowledge of the coefficients provides, for example, a way to calculate the zeta-functions for these manifolds. This is an interesting topic to which we will return elsewhere.

The periods of a Calabi–Yau manifold play an essential role in the description of the manifold and of its parameter space. The periods, in turn, satisfy Picard-Fuchs equations. The question arises as to which differential operators are associated with the parameter spaces of Calabi–Yau manifolds. Almqvist *et al.* [11] have formulated conditions that set out when a differential operator is of ‘Calabi–Yau type’ and have assembled a table of operators that satisfy the conditions. If these conditions are satisfied, then the yukawa coupling, computed from the periods, has a q -expansion

$$y = y_0 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1 - q^k}$$

with integer coefficients y_0 and n_k . In some cases the differential equation and the periods are known to correspond to a Calabi–Yau manifold. In other cases this is not known. It was observed by Guttman [12] and by Broadhurst [13] that certain generating functions for lattice walks satisfy differential equations of Calabi–Yau type and two of these functions and equations correspond to the manifolds that we discuss here. These differential equations are listed in the table of [11]. Thus the differential equations were already known and the expansions for the yukawa couplings were known up to an overall multiplicative integer. Here we derive the periods and the differential equation from the manifolds in question and study the monodromies of the periods around the singularities of the moduli space. The observations that we make here about the occurrence of the quantity $\zeta(3)/(2\pi i)^3$ in the matrix that relates the Frobenius basis of the periods to the integral basis, and in the monodromy matrices for the Frobenius basis, could have been made already in relation to the quintic Calabi–Yau manifold. In the context of the monodromy matrices for the operators of Calabi–Yau type, the ubiquity of $\zeta(3)/(2\pi i)^3$ was noted in the extensive calculations of Hoffman [14].

2. The Special Geometry of the (4,1)-Manifold

2.1. Fundamental Period

From [8] we have that the fundamental period is of the form discussed in the Introduction. It is given by the following integral representation

$$\varpi_0(\varphi) = \frac{1}{(2\pi i)^4} \int \frac{d^4 t}{t_1 t_2 t_3 t_4 (1 - \varphi f(t))}$$

where

$$f(t) = T(t_1, t_2) + T(t_3, t_4) \quad \text{with} \quad T(t_1, t_2) = t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + \frac{t_1}{t_2} + \frac{t_2}{t_1}, \quad (2.1)$$

and the contour, for the integral, consists of a product of four loops enclosing the poles $t_j = 0$. From the integral we see that

$$\varpi_0(\varphi) = \sum_{n=0}^{\infty} \varphi^n [f(t)^n]_{t^0},$$

with $[f(t)^n]_{t^0}$ denoting the constant term in the expansion of $f(t)^n$. In order to compute the coefficients $[f(t)^n]_{t^0}$ efficiently we set $T_k = [T(t_1, t_2)^k]_{t^0}$ and note that

$$[f(t)^n]_{t^0} = \sum_{r=0}^n \binom{n}{r} T_{n-r} T_r = \begin{cases} 2 \sum_{r=0}^{\frac{1}{2}(n-1)} \binom{n}{r} T_{n-r} T_r ; & \text{if } n \text{ is odd} \\ 2 \sum_{r=0}^{\frac{n}{2}-1} \binom{n}{r} T_{n-r} T_r + \binom{n}{\frac{n}{2}} T_{\frac{n}{2}}^2 ; & \text{if } n \text{ is even .} \end{cases}$$

We can calculate the quantities T_k explicitly by collecting powers of t_1

$$T(t_1, t_2) = t_1 \left(1 + \frac{1}{t_2}\right) + \left(t_2 + \frac{1}{t_2}\right) + \frac{1}{t_1}(1 + t_2)$$

and picking out the terms in the expansion of $T(t_1, t_2)^k$ that are independent of t_1 and then of t_2 . In this way we find

$$T_k = \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{s=\max(0, \lceil \frac{k-3r}{2} \rceil)}^{\min(k-2r, \lfloor \frac{k-r}{2} \rfloor)} \frac{k! (2r)!}{(r!)^2 s! (3r+2s-k)! (k-r-2s)! (k-2r-s)!} . \quad (2.2)$$

It is easy, proceeding in this way, to compute several hundred terms, say, of the series. The first few terms in the expansion of ϖ_0 are given by

$$\begin{aligned} \varpi_0(\varphi) = & 1 + 12 \varphi^2 + 24 \varphi^3 + 396 \varphi^4 + 2160 \varphi^5 + 23160 \varphi^6 + 186480 \varphi^7 \\ & + 1845900 \varphi^8 + 17213280 \varphi^9 + 171575712 \varphi^{10} + O(\varphi^{11}) \end{aligned} \quad (2.3)$$

In terms of lattices, the 12 vertices of the dual polytope arrange into the nearest neighbours of two copies of the two-dimensional hexagonal (a.k.a. triangular) lattice. This much is clear from the description of the $(4, 1)$ -manifold as a hypersurface in the product of two toric dP_6 surfaces. The T_k count closed length- k walks in the hexagonal lattice. There is no closed form (not involving a double sum) for the coefficients, but we cannot resist remarking that there are interesting expressions that are known ordinary and exponential generating functions [15]:

$$\begin{aligned}\sum_k T_k x^k &= \sum_k \frac{(3k)!}{k!^3} x^{2k} (1+2x)^k = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 27x^2(1+2x)\right), \\ \sum_k \frac{T_k}{k!} x^k &= I_0(2x)^3 + 2 \sum_{k=1}^{\infty} I_k(2x)^3.\end{aligned}$$

Where ${}_2F_1$ denotes the hypergeometric function and the I_k are Bessel functions of imaginary argument. The exponential generating function is multiplicative over lattice products, hence we can write the exponential generating function of the fundamental period using Bessel functions as

$$E_{\varpi_0}(\varphi) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \varphi^n = \left(I_0(2\varphi)^3 + 2 \sum_{k=1}^{\infty} I_k(2\varphi)^3 \right)^2.$$

Another way of thinking about the closed walks in the hexagonal lattice is as closed paths where the steps are sixth roots of unity. In fact, a walk whose steps are *twelfth* root of unity closes if and only if the even and odd powers of the fundamental root close separately. So, if β is a primitive twelfth root of unity, we have

$$\sum_{j=0}^{11} k_j \beta^j = 0 \quad \Leftrightarrow \quad \sum_{j=0}^5 k_{2j} \beta^{2j} = 0 \quad \text{and} \quad \sum_{j=0}^5 k_{2j+1} \beta^{2j+1} = 0.$$

Therefore, we can also identify the fundamental period as the (ordinary) generating functions of closed walks using 12-th roots of unity [16], leading to the more symmetric formula

$$\varpi_0(\varphi) = \sum_{\substack{k_0 + \dots + k_{11} = n \\ k_0 k_{10} = k_4 k_6 \\ k_1 k_{11} = k_5 k_7 \\ k_2 k_4 = k_8 k_{10} \\ k_3 k_5 = k_9 k_{11}}} \frac{n!}{k_0! k_1! \dots k_{11}!} \varphi^n$$

2.2. Picard-Fuchs Differential Equation

We seek a fourth order differential equation that the fundamental period satisfies, as indicated in the Introduction, and increase k_{\max} until we come to $k_{\max} = 8$. This process yields the following coefficient functions R_j :

$$\begin{aligned}
R_4 &= 69120 \left(\varphi - \frac{3}{2}\right)^2 \left(\varphi + \frac{1}{4}\right) \left(\varphi + \frac{1}{5}\right) \left(\varphi + \frac{1}{6}\right) \left(\varphi - \frac{1}{12}\right) \left(\varphi - \frac{1}{4}\right) \left(\varphi - \frac{1}{3}\right) \\
R_3 &= 8\varphi \left(\varphi - \frac{3}{2}\right) (86400\varphi^6 - 158976\varphi^5 - 1512\varphi^4 + 15964\varphi^3 + 160\varphi^2 - 345\varphi - 5) \\
R_2 &= \varphi (2419200\varphi^7 - 8581248\varphi^6 + 7771104\varphi^5 + 274360\varphi^4 - 552220\varphi^3 - 5250\varphi^2 \\
&\quad + 6917\varphi + 39) \\
R_1 &= \varphi (3456000\varphi^7 - 12745728\varphi^6 + 12372480\varphi^5 + 166288\varphi^4 - 679952\varphi^3 - 1584\varphi^2 \\
&\quad + 5532\varphi + 9) \\
R_0 &= 48\varphi^2 (34560\varphi^6 - 130464\varphi^5 + 132120\varphi^4 + 284\varphi^3 - 6182\varphi^2 + 9\varphi + 36)
\end{aligned}$$

Notice that, apart from the factor of $(\varphi - \frac{3}{2})^2$, we recognise the factors of R_4 from Table 7 of [8]. From the ratio R_3/R_4 we may compute the Yukawa coupling

$$\begin{aligned}
y_{\varphi\varphi\varphi} &= \frac{1}{\varphi^3} \exp \left(-\frac{1}{2} \int \frac{d\varphi}{\varphi} \frac{R_3(\varphi)}{R_4(\varphi)} \right) \\
&= \text{const.} \times \frac{\varphi - \frac{3}{2}}{\varphi^3 \left(\varphi + \frac{1}{4}\right) \left(\varphi + \frac{1}{5}\right) \left(\varphi + \frac{1}{6}\right) \left(\varphi - \frac{1}{12}\right) \left(\varphi - \frac{1}{4}\right) \left(\varphi - \frac{1}{3}\right)}.
\end{aligned}$$

Notice that the coupling has a zero at $\varphi = \frac{3}{2}$.

Since we now have the differential equation we may compute the indices at the singular points. These are summarised by the Riemann symbol, which records the orders of the vanishing of the four solutions of the Picard–Fuchs equation at the singularities of the equation.

$$\mathcal{P} \left\{ \begin{array}{ccccccccc} 0 & \infty & \frac{3}{2} & -\frac{1}{4} & -\frac{1}{5} & -\frac{1}{6} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \end{array} \right. \varphi$$

Note first that all four indices at $\varphi=0$ vanish so this is a point of maximal unipotent monodromy. This confirms our identification of this point as the large complex structure limit. The method of Frobenius yields a basis of solutions for the operator \mathcal{L} as in (1.1). The procedure is readily adapted to computer calculation. First, given \mathcal{L} , we solve the equation

$\mathcal{L}\varpi_0 = 0$ for the coefficients of the series h_0 , then we solve the equation $\mathcal{L}\varpi_1 = 0$ for the coefficients of the series h_1 , and so on.

We form the periods ϖ_j into a vector, as in (1.2), and consider the monodromy $\varpi \rightarrow m_0 \varpi$ going anti-clockwise in a small circle about the origin. This monodromy is generated by $\log \varphi \rightarrow \log \varphi + 2\pi i$, and in this way we see that

$$m_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

Consider next the monodromy about one of the six points that correspond to the indices $\{0, 1, 1, 2\}$ which, given the indices and the type of singularity recorded in Table 7 of [8], we expect to be of conifold type. That is we expect m to have the form

$$m = \mathbb{1} + a ,$$

with $a^2 = 0$ so that $m^n = \mathbb{1} + na$. We can say more, since we expect a to be a matrix of rank 1 and therefore to have the structure

$$a_{ij} = u_i v_j \quad \text{with} \quad u_j v_j = 0 ,$$

the latter condition following from the fact that $a^2 = 0$. Mathematica provides a function that, given suitable initial conditions, integrates a differential equation along a line segment and returns an interpolating function. Four such line segments may be concatenated to form a rectangle. So, starting from a point near $\varphi = 0$, where the values of ϖ and its derivatives may be computed from (2.3), we may integrate the Picard-Fuchs equation around a rectangle that encloses some of the singular points in order to find the monodromy matrices.

Consider the monodromy matrix that corresponds to the singularity at $\varphi = 1/12$. The numerical procedure just described returns the matrix²

$$m_{\frac{1}{12}} - \mathbb{1} = \begin{pmatrix} 0.348914i & -6 & 0 & -36 \\ 0 & 0 & 0 & 0 \\ 0.0193841i & -0.333333 & 0 & -2 \\ -0.00338169 & -0.0581523i & 0 & -0.348914i \end{pmatrix} .$$

Let us denote the $(1, 1)$ -component of the matrix above by $i\alpha$. We see that the matrix has indeed the form $u_i v_j$ since the second, third and fourth rows are obtained from the first by multiplying by 0, $1/18$ and $i\alpha/36$, respectively. Let us define now two antisymmetric matrices which prove useful in the following

$$\Sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = \frac{1}{Y_{111}} \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{pmatrix} , \quad (2.4)$$

²It is straightforward to perform the computations to 70 figures, say. But we show a simplified output.

where we write an explicit factor of the classical yukawa coupling (=18 for this manifold) in our definition of σ to show the dependence on the yukawa coupling for different manifolds. We find, in fact, that $m_{\frac{1}{12}}$ has the form

$$m_{\frac{1}{12}} = \mathbb{1} - 12 u u^T \sigma^{-1} \quad \text{with} \quad u = \begin{pmatrix} 1 \\ 0 \\ 1/18 \\ i\alpha/36 \end{pmatrix}.$$

The prepotential contains a term $i\zeta(3)/(2\pi)^3$ and this suggests that we try to identify the number α in terms of this. We find

$$\alpha = 72 \frac{\zeta(3)}{(2\pi)^3}.$$

The monodromies about the other conifold points have an analogous structure. In each case the matrix takes the form

$$m_{ij} = \delta_{ij} - \mu u_i (u^T \sigma^{-1})_j. \quad (2.5)$$

We record the coefficients μ and vectors u in Table 1. The points $\varphi = 3/2$ and $\varphi = \infty$ are singularities of the Picard-Fuchs equation though the manifold is smooth for these values. Computation reveals that the monodromy around these points is trivial

$$m_{\frac{3}{2}} = m_{\infty} = \mathbb{1}.$$

The fact that $m_{\frac{3}{2}}$ is trivial is checked by direct computation. The triviality of the monodromy about ∞ is then a consequence of the fact that the total monodromy around all other singularities is trivial and this amounts to the identity

$$m_{-\frac{1}{4}} m_{-\frac{1}{5}} m_{-\frac{1}{6}} m_0 m_{\frac{1}{12}} m_{\frac{1}{4}} m_{\frac{1}{3}} = \mathbb{1}.$$

Owing to the monodromy the periods ϖ_j are multiple valued functions. We may render them single valued by cutting the φ -plane from $\varphi = -1/3$ to $\varphi = 1/4$ and taking the expressions (2.3) to hold in the region $|\varphi| < 1/12$, $\text{im}(\varphi) \geq 0$.

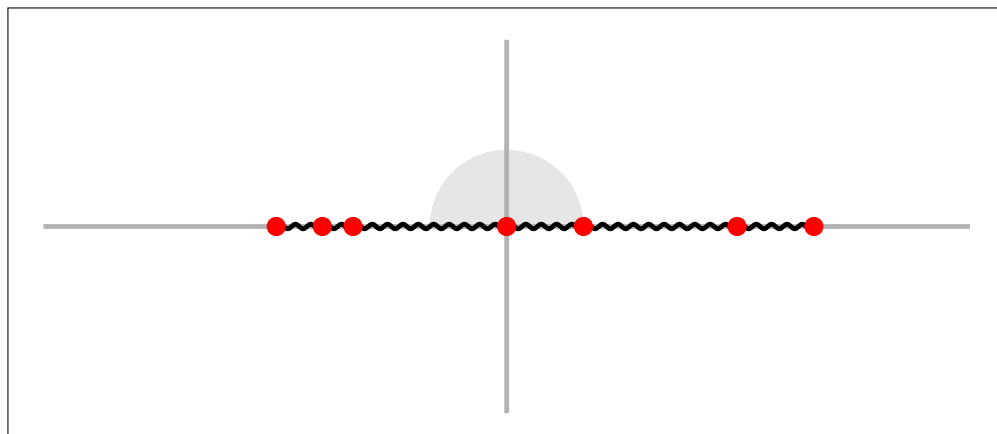


Figure 1: The φ -plane cut from $-1/4$ to $1/3$ and showing the singularities of $\varpi(\varphi)$ at $\varphi = -\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}, 0, \frac{1}{12}, \frac{1}{4}, \frac{1}{3}$. The shaded area corresponds the region where the expression (2.3) holds.

φ	number	singularity	μ	u^T	U^T
$-\frac{1}{4}$	(ii), (x)	$1\,g_4, 1\,g_4^2$	12	$(3, 1, \frac{7}{18}, \frac{1}{6} + \frac{i\alpha}{12})$	(1, -2, 3, 1)
$-\frac{1}{5}$	(vi)	1	1	$(12, 5, \frac{7}{3}, \frac{7}{6} + \frac{i\alpha}{3})$	(6, -15, 12, 5)
$-\frac{1}{6}$	(iv)	$1\,g_3$	3	$(4, 2, \frac{10}{9}, \frac{2}{3} + \frac{i\alpha}{9})$	(3, -8, 4, 2)
$\frac{1}{12}$	(iii)	$1\,\mathfrak{G}$	12	$(1, 0, \frac{1}{18}, \frac{i\alpha}{36})$	(0, 0, 1, 0)
$\frac{1}{4}$	(ix)	$1\,g_4^2$	2	$(6, 1, \frac{1}{3}, \frac{1}{6} + \frac{i\alpha}{6})$	(1, 0, 6, 1)
$\frac{1}{3}$	(v)	$1\,g_3$	3	$(4, 1, \frac{1}{3}, \frac{1}{6} + \frac{i\alpha}{9})$	(1, -1, 4, 1)

Table 1: The vectors u corresponding to the monodromy matrices for each of the hyperconifold points. The vectors U are the same vector in the integral symplectic homology basis, see Section 2.3. The “number” column refers to the labelling of the components of the discriminant locus given in [8] and the “singularity” column records the number and type of the hyperconifold singularities [17]. Thus $1\,\mathfrak{G}$, for example, refers to a single hyperconifold point that is the \mathfrak{G} -quotient of a node, while $1\,g_4, 1\,g_4^2$, refers to one hyperconifold that is a g_4 -quotient and one that is a g_4^2 -quotient.

2.3. Integral Homology Basis and the Prepotential

We turn now to the task of finding an integral basis for the homology and the prepotential, \mathcal{F} , for the mirror of $Y_{\mathfrak{G}}$. For the perturbative part of the prepotential we take

$$\mathcal{F}^{\text{pert}} = -\frac{1}{3!} Y_{abc} \frac{z^a z^b z^c}{z^0} \quad \text{with} \quad Y_{111} = 18 ,$$

which is written in terms of the special coordinates (z^0, z^1) and the value given for Y_{111} follows from an intersection calculation.

In general, when there are n parameters, the indices run over the range $a, b, c = 0, \dots, n$. For $i, j, k = 1, \dots, n$ the Y_{ijk} are the topological intersection numbers

$$Y_{ijk} = \int_{\mathcal{M}} e_i e_j e_k ,$$

with the e_i a basis for $H^2(\mathcal{M}, \mathbb{Z})$. The remaining coefficients Y_{0jk} , Y_{00k} and Y_{000} depend on a

choice of symplectic basis for $H^3(\mathcal{W}, \mathbb{Z})$. In the following we will make reference to the Chern classes of \mathcal{M} which we denote by c_r . The term Y_{000} is believed to be related to the four loop term for the β -function for $(2, 2)$ supersymmetric sigma-models on Calabi–Yau threefolds and to be, in all cases, such that by means of an integral symplectic transformation it may be brought to the form

$$Y_{000} = -3 \frac{\zeta(3)}{(2\pi i)^3} \chi(X) . \quad (2.6)$$

Where $\chi = \int c_3$ denotes the Euler number. It has been observed also that the terms Y_{00k} can also be brought, by means of an $\mathrm{Sp}(n+1, \mathbb{Z})$ transformation, to the form

$$Y_{00k} = -\frac{1}{12} \int_{\mathcal{M}} c_2 e_k . \quad (2.7)$$

Given the last two relations it would seem intuitive that the terms Y_{0jk} can be brought to a form proportional to

$$\int_{\mathcal{M}} c_1 e_j e_k \quad (2.8)$$

and hence be set to zero. This cannot, however, be true in all cases since the prepotential for the mirror quintic furnishes a counterexample. In that case the quantity Y_{011} takes values that are half an odd integer and the values can only be changed by an integer, by an $\mathrm{Sp}(4, \mathbb{Z})$ transformation. For the manifold that we are considering here we will, nevertheless, see that it is possible to take $Y_{011} = 0$.

From the full prepotential, \mathcal{F} , we form the period vector

$$\Pi = \begin{pmatrix} \frac{\partial \mathcal{F}}{\partial z^0} \\ \frac{\partial \mathcal{F}}{\partial z^1} \\ z^0 \\ z^1 \end{pmatrix} .$$

Both Π and ϖ are vectors of periods so are related by a matrix of constants ρ

$$\Pi = \rho \varpi .$$

We identify $(z^0, z^1) = (\varpi_0, \varpi_1)$ and set

$$t = \frac{z^1}{z^0} = \frac{1}{2\pi i} \log \varphi + \frac{h_1(\varphi)}{h_0(\varphi)} . \quad (2.9)$$

The series h_1 has been defined such that $h_1(0) = 0$, so as $\varphi \rightarrow 0$ we have $t = \frac{1}{2\pi i} \log \varphi + \mathcal{O}(\varphi)$. The asymptotic form of the period vectors, as $\varphi \rightarrow 0$ is enough to relate ρ to the constants that appear $\mathcal{F}^{\mathrm{pert}}$. We have

$$\Pi \sim \begin{pmatrix} \frac{1}{6} t^3 Y_{111} - \frac{1}{2} t Y_{001} - \frac{1}{3} Y_{000} \\ -\frac{1}{2} t^2 Y_{111} - t Y_{011} - \frac{1}{2} Y_{001} \\ 1 \\ t \end{pmatrix} \quad \text{and} \quad \varpi \sim \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} .$$

Thus

$$\rho = \begin{pmatrix} -\frac{1}{3}Y_{000} & -\frac{1}{2}Y_{001} & 0 & \frac{1}{6}Y_{111} \\ -\frac{1}{2}Y_{001} & -Y_{011} & -\frac{1}{2}Y_{111} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Every matrix ρ , of this form, has the property that

$$\rho\sigma\rho^T = \Sigma.$$

Note also that any monodromy matrix of the form (2.5) preserves the matrix σ

$$m\sigma m^T = \sigma.$$

In the Π - basis we have monodromy matrices

$$M = \rho m \rho^{-1} = \mathbb{1} + \mu U U^T \Sigma \quad \text{with} \quad U = \rho u \quad (2.10)$$

and these satisfy

$$M \Sigma M^T = \Sigma.$$

Thus the matrices M are symplectic. We need to check that they are also integral. The matrix $M_{\frac{1}{12}}$ has, among its components, the quantities $(i\alpha - 4Y_{000})$ and $\frac{4}{3}(i\alpha - 4Y_{000})^2$. For these to be integral we must have

$$Y_{000} = \frac{i\alpha}{4} + \frac{3}{2}C \quad \text{with} \quad C \in \mathbb{Z}.$$

The integrality of M_0 requires both Y_{001} and Y_{011} to be integral. Another constraint comes from $M_{\frac{1}{3}}$, which contains the term $\frac{3}{2}(Y_{001} - 1 + 4C)$. Thus

$$Y_{001} = -1 + 2B \quad \text{with} \quad B \in \mathbb{Z}.$$

For uniformity we write $Y_{011} = A$. With A , B and C integral all the matrices M are integral.

The matrix

$$S = \begin{pmatrix} 1 & 0 & C' & B' \\ 0 & 1 & B' & A' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is integral and symplectic if A' , B' and C' are integral and

$$S\rho = \begin{pmatrix} \frac{i\alpha}{12} - \frac{1}{2}C + C' & \frac{1}{2} - A + A' & 0 & 3 \\ \frac{1}{2} - A + A' & -B + B' & -9 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The upshot is that, by means of an $\text{Sp}(4, \mathbb{Z})$ transformation that, moreover, preserves the coordinates (z^0, z^1) , we may set $A = B = 0$ and also remove the integer part of $C/2$.

Although it does not follow from the integrality of M we will nevertheless assume that C is even so that it may be removed by a symplectic transformation in conformity with our previous observations concerning Y_{000} . Thus we take ρ to be given by

$$\rho = \begin{pmatrix} -\frac{i\alpha}{12} & \frac{1}{2} & 0 & 3 \\ \frac{1}{2} & 0 & -9 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In summary we may take

$$Y_{111} = 18, \quad Y_{011} = 0, \quad Y_{001} = -1, \quad Y_{000} = 18 \frac{\zeta(3)}{(2\pi i)^3}.$$

For our manifold we have $\chi(Y_{\mathfrak{G}}) = -6$ and $\int_{Y_{\mathfrak{G}}} c_2 e_1 = 12$, so these values for the constants are in conformity with the forms (2.6), (2.7) and (2.8). With these values for the constants the perturbative part of the superpotential assumes the very simple form

$$\mathcal{F}^{\text{pert}} = -3 \frac{(z^1)^3}{z^0} + \frac{1}{2} z^0 z^1 - 3 \frac{\zeta(3)}{(2\pi i)^3} (z^0)^2.$$

Let us return to the form of the monodromy matrices. In writing the matrices in the form (2.10) there is a freedom to transfer a numerical factor between the coefficient μ and the vector U . We fix the normalization of U , up to sign, by requiring that the components of U be integral and have no common factor. The coefficients and vectors that are presented in Table 1 are normalised in this way. Note that, with the possible exception of the monodromy about $\varphi = -1/4$, the coefficients μ correspond to the ‘equivalent number of conifolds’ with a \mathfrak{G} -hyperconifold counting as 12 conifold points and a g_3 hyperconifold counting as 3. The singularity for $\varphi = -1/4$ corresponds to a g_4 -hyperconifold and a g_4^2 -hyperconifold so, from this point of view, ought to count as six conifold points instead of 12. The singularity at $\varphi = -1/4$ is, however, special in that this singularity corresponds to the intersection of the φ -plane with the two components (ii) and (x) of the discriminant locus. The component (ii) corresponds to the hyperconifold singularities while the component (x) is fixed by a possible \mathbb{Z}_2 automorphism of the two parameter space Γ .

We have determined the matrix ρ , hence Π . The special geometry of the complex structures of the mirror of $Y_{\mathfrak{G}}$ is now calculable in a straightforward way. The Kähler potential of the metric on the parameter space, for example, is given by

$$e^{-K} = -i \Pi^\dagger \Sigma \Pi = -i \varpi^\dagger \rho^\dagger \Sigma \rho \varpi = \frac{12 \zeta(3)}{(2\pi)^3} |\varpi_0|^2 + i \varpi^\dagger \sigma^{-1} \varpi.$$

Since the complex structure space is one-dimensional, there is only a single nonvanishing metric component $g_{\varphi\bar{\varphi}}$. We plot its behavior in Figure 2.

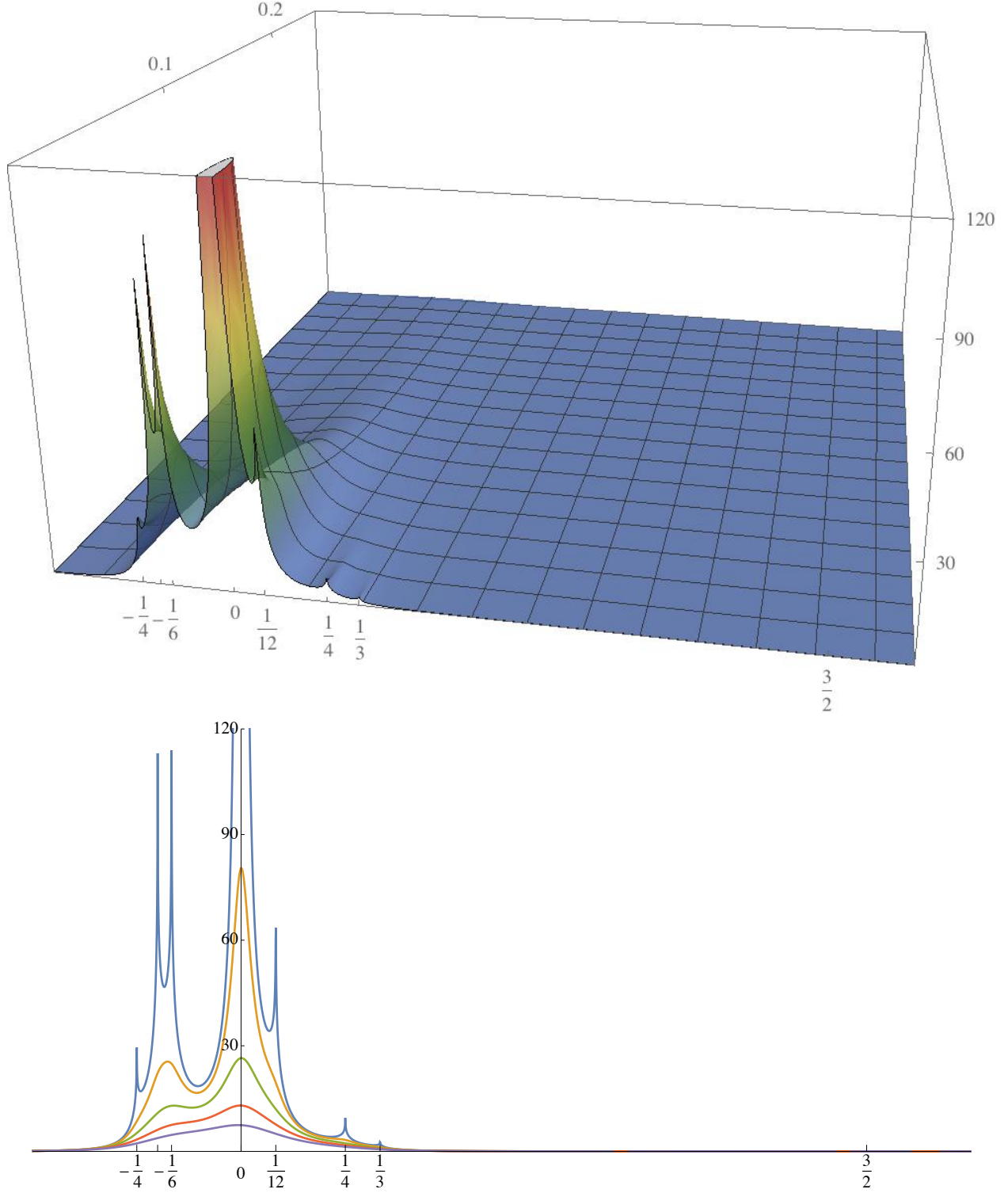


Figure 2: The metric $g_{\varphi\bar{\varphi}}$ on the complex structure moduli space. Top: plot over a region of the complex plane. Bottom: restriction to a real interval (blue) and its shift to constant imaginary value (remaining colors).

2.4. The Yukawa coupling

The yukawa coupling, in the symplectic basis, is given by the expression

$$y_{ttt} = \frac{1}{1440 \varpi_0(\varphi)^2} \frac{(\varphi - \frac{3}{2})}{\varphi^3 (\varphi - \frac{1}{3}) (\varphi - \frac{1}{4}) (\varphi - \frac{1}{12}) (\varphi + \frac{1}{4}) (\varphi + \frac{1}{5}) (\varphi + \frac{1}{6})} \left(\frac{1}{2\pi i} \frac{d\varphi}{dt} \right)^3 ,$$

an overall scale having been chosen such that $y_{ttt} \rightarrow 18$ as $\varphi \rightarrow 0$. We invert the relation (2.9) to write φ as a series in the variable $q = e^{2\pi i t}$.

$$\varphi = q - q^2 - 14 q^3 + 9 q^4 + 146 q^5 - 406 q^6 - 2967 q^7 + 2055 q^8 + 9100 q^9 + \dots$$

In this way we obtain a series expansion for y_{ttt} which has the form of an instanton expansion

$$y_{ttt} = 18 + \sum_{k=0}^{\infty} \frac{n_k k^3 q^k}{1 - q^k} ,$$

with the first few instanton numbers given in the table.

k	n_k	k	n_k	k	n_k
1	6	11	15261150	21	3721663648494978
2	15	12	97808574	22	27257992426100979
3	30	13	641284110	23	201015705767041110
4	114	14	4287838548	24	1491738880927589808
5	522	15	29153498904	25	11134231701698352462
6	2529	16	201163103922	26	83548054037756075397
7	12636	17	1406107987374	27	630009551885176580298
8	69744	18	9941935540692	28	4772408720293409689260
9	405168	19	71017384630734	29	36305219849933449704906
10	2449773	20	511976000663130	30	277278864871401475773546

Table 2: Instanton numbers of the $(4,1)$ -threefold.

3. The Special Geometry of the (1,1)-Manifold(s)

3.1. The Picard-Fuchs Operator

The most symmetric 4-dimensional reflexive polytope is the 24-cell. It admits three order-24 group actions that permute the 24 vertices simply transitively such that there are no fixed points on the Calabi-Yau hypersurface, hence defining 3 different manifolds with Hodge numbers $h^{1,1} = h^{2,1} = 1$. For a full description see [10]. In each of these three cases, the Calabi-Yau hypersurface in the covering space toric variety is again of the form

$$P_\varphi = \prod_{i=1}^{24} z_i - \varphi \sum_{i=1}^{24} z^{v_i} = t_1 t_2 t_3 t_4 - \varphi \sum_{i=1}^{24} t^{v_i} = t_1 t_2 t_3 t_4 (1 - \varphi f(t)),$$

written either in homogeneous coordinates $\{z_i\}_{i=1,\dots,24}$ or in inhomogeneous coordinates t_1, \dots, t_4 . So we have to again compute $[f^n(t)]_0$. This time, the Laurent polynomial is

$$\begin{aligned} f(t) &= t_1 + t_2 + t_3 + t_4 + \frac{t_1 t_4}{t_2 t_3} + \frac{t_4}{t_3} + \frac{t_4}{t_2} + \frac{t_4}{t_1} + \frac{t_1}{t_4} + \frac{t_2}{t_4} + \frac{t_3}{t_4} + \frac{t_2 t_3}{t_1 t_4} + \frac{t_1}{t_2 t_3} \\ &\quad + \frac{1}{t_3} + \frac{1}{t_2} + \frac{1}{t_1} + \frac{t_1}{t_2} + \frac{t_1}{t_3} + \frac{t_2 t_3}{t_4} + \frac{t_2 t_3}{t_1} + \frac{t_2}{t_1} + \frac{t_3}{t_1} + \frac{t_4}{t_2 t_3} + \frac{1}{t_4} \\ &= T(t_1, t_2) + T(t_3, t_4) + T\left(\frac{t_1}{t_3}, \frac{t_2}{t_4}\right) + T\left(\frac{t_1}{t_4}, \frac{t_2 t_3}{t_4}\right), \end{aligned} \quad (3.1)$$

where we used the same expression for T as before in eq. (2.1). On this occasion, factors originating from different T -summands can cancel in powers f^n , so we need the full expansion for the powers of $T(t_1, t_2)$

$$T(t_1, t_2)^k = \sum_{\alpha, \beta = -k}^k T_{k\alpha\beta} t_1^\alpha t_2^\beta.$$

Explicitly, we have

$$\begin{aligned} T_{k\alpha\beta} &= \sum_{\rho = \max(\alpha, 0)}^{\lfloor \frac{k+\alpha}{2} \rfloor} \sum_{j = \max(\lceil \frac{k+2\alpha+\beta-3\rho}{2} \rceil, 0)}^{\min(k+\alpha-2\rho, \lfloor \frac{k+\alpha+\beta-\rho}{2} \rfloor)} \frac{(2\rho - \alpha)!}{\rho! (\rho - \alpha)!} \times \\ &\quad \times \frac{k!}{j! (3\rho + 2j - k - 2\alpha - \beta)! (k + \alpha + \beta - \rho - 2j)! (k + \alpha - 2\rho - j)!}. \end{aligned}$$

The powers of the polynomial are

$$f(t)^n = \sum_{\substack{a, b, c, d \geq 0 \\ a+b+c+d=n}} \frac{n!}{a! b! c! d!} T^a(t_1, t_2) T^b(t_3, t_4) T^c\left(\frac{t_1}{t_3}, \frac{t_2}{t_4}\right) T^d\left(\frac{t_1}{t_4}, \frac{t_2 t_3}{t_4}\right)$$

On substituting the expansions for the T 's and picking out the terms independent of the t_i , we obtain the fundamental period as

$$\varpi_0(\varphi) = \sum_{n=0}^{\infty} \sum_{\substack{a,b,c,d \geq 0 \\ a+b+c+d=n}} \frac{n!}{a!b!c!d!} \varphi^n \sum_{\alpha,\beta=-a}^a \sum_{\gamma,\delta=-b}^b T_{a\alpha\beta} T_{b\gamma\delta} T_{c,\alpha+\beta+\delta,-(\alpha+\gamma+\delta)} T_{d,\beta+\delta,-(\alpha+\beta+\gamma+\delta)}.$$

This expression is somewhat involved but presents little difficulty to machine computation.

The fundamental period again must obey a fourth-order differential equation of the form (1.3). Expanding ϖ_0 to sufficiently high order and solving for polynomials R_j yields degree-7 coefficients

$$\begin{aligned} R_4 &= 8957952 \left(\varphi + \frac{1}{18}\right)^2 \left(\varphi + \frac{1}{3}\right) \left(\varphi + \frac{1}{4}\right) \left(\varphi + \frac{1}{8}\right) \left(\varphi + \frac{1}{12}\right) \left(\varphi - \frac{1}{24}\right) \\ R_3 &= 36\varphi \left(\varphi + \frac{1}{18}\right) (1990656\varphi^5 + 1257984\varphi^4 + 264384\varphi^3 + 22320\varphi^2 + 800\varphi + 15) \\ R_2 &= \varphi(206032896\varphi^6 + 118195200\varphi^5 + 24103872\varphi^4 + 2276640\varphi^3 + 105552\varphi^2 \\ &\quad + 2114\varphi + 19) \\ R_1 &= 72\varphi \left(\varphi + \frac{1}{18}\right) (3483648\varphi^5 + 1548288\varphi^4 + 225072\varphi^3 + 13572\varphi^2 + 320\varphi + 1) \\ R_0 &= 96\varphi^2 (1119744\varphi^5 + 508032\varphi^4 + 82512\varphi^3 + 6318\varphi^2 + 237\varphi + 4) . \end{aligned}$$

As already noted, this Picard-Fuchs equation has been found previously [11, 12] in a non-geometric context, just from studying the generating function for walks in the 4-d face-centered³ hyper-cubic (fcc) lattice. We will explore the connection with walks in detail in Section 4. For now note that the naive generators of this lattice are

$$\{(2, 0, 0, 0), \dots, (0, 0, 0, 2), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}, \quad (3.2)$$

and the nearest neighbors of the origin indeed span a polytope that is $\text{GL}(4, \mathbb{Q})$ -equivalent to the reflexive 24-cell. However, it has twice the lattice volume, hence it is not $\text{SL}(4, \mathbb{Q})$ -equivalent. The fcc nearest neighbor polytope, that is, the polytope whose vertices are all permutations and sign flips of $(1, 1, 0, 0)$, is not reflexive and we cannot associate a toric Calabi-Yau hypersurface to it. However, it is combinatorially the same as the lattice generated by the vertices of the lattice 24-cell. In fact, the presentation as the fcc lattice is easier, and [13, 18] give the nicer generating series

$$\begin{aligned} \varpi_0(\varphi) &= \sum_{i,j,k,l,m=0}^{\infty} \binom{2i}{i} \binom{2j}{j} \binom{2k}{k} \binom{l+m}{m} \binom{2(l+m)}{l+m}^2 \times \\ &\quad \times \binom{i+j+k+l+m}{2(l+m)} \binom{i+j+k-l-m}{-i+j+k} \binom{2i-l-m}{i-k-l} \varphi^{i+j+k+l+m}. \end{aligned} \quad (3.3)$$

³That is, 2-face centered: there is an extra lattice point in the center of each 2-face of the simple cubic lattice. Note that, in 4-d, these automatically generate a lattice point in the center $(1, 1, 1, 1)$ of the cubic cell.

The full set of solutions to the Picard Fuchs equation is as in (1.1). With the series h_j given by

$$\begin{aligned}
h_0 &= 1 + 24\varphi^2 + 192\varphi^3 + 3384\varphi^4 + 51840\varphi^5 + 911040\varphi^6 + \mathcal{O}(\varphi^7) \\
h_1 &= 4\varphi + 40\varphi^2 + \frac{1816}{3}\varphi^3 + 8644\varphi^4 + \frac{753744}{5}\varphi^5 + \frac{7747372}{3}\varphi^6 + \mathcal{O}(\varphi^7) \\
h_2 &= -6\varphi - \frac{51}{2}\varphi^2 - \frac{3062}{3}\varphi^3 - \frac{299593}{24}\varphi^4 - \frac{19639868}{75}\varphi^5 - \frac{44274137}{10}\varphi^6 + \mathcal{O}(\varphi^7) \\
h_3 &= -36\varphi - \frac{57}{2}\varphi^2 - \frac{6346}{3}\varphi^3 - \frac{231785}{16}\varphi^4 - \frac{71264197}{250}\varphi^5 - \frac{821688758}{225}\varphi^6 + \mathcal{O}(\varphi^7)
\end{aligned}$$

The singularities of the Picard-Fuchs equation are evident from the factorization of R_4 . The behavior of solutions around each of the singularities is summarized by the Riemann symbol

$$\mathcal{P} \left\{ \begin{array}{cccccccc} 0 & \infty & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{12} & -\frac{1}{18} & \frac{1}{24} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 3 & 1 \\ 0 & 3 & 2 & 2 & 2 & 2 & 4 & 2 \end{array} \varphi \right\}, \quad (3.4)$$

where we again notice the point of maximal unipotent monodromy at $\varphi = 0$ and a point with four power series solutions (and, therefore, no monodromy) at $\varphi = -\frac{1}{18}$. We will see that $\varphi = \infty$ corresponds to a conifold point. This suggests a change of variable that brings this conifold to a finite point and so strengthens the similarity to the situation for the (1,4)-manifold. We will return to this after considering the various monodromy matrices.

3.2. Monodromy

The geometric meaning of the singularities requires us to look at the details of the group actions when descending to the $h^{11} = h^{21} = 1$ quotient, and is summarized in Table 3.

Owing to the logarithms the period vector ϖ transform as $\varpi \mapsto m_0 \varpi$, under monodromy around $\varphi = 0$ with the same monodromy matrix m_0 as in eq. (2.2).

Computing the monodromies numerically around the other singularities, one finds a rational matrix m_φ for each. Apart from the large complex structure $\varphi = 0$ and the point $\varphi = -\frac{1}{18}$, which (surprisingly) corresponds to a smooth manifold, these all have a single logarithmic period. Hence, we expect them to be of conifold type again and they should be expressible as

$$m_\varphi = \mathbb{1} - \mu_\varphi u_\varphi u_\varphi^T \sigma^{-1} \quad (3.5)$$

where

φ	$\tilde{\mu}(P_\varphi)$	$G_1 \simeq \text{SL}(2, 3)$		$G_2 \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_8$		$G_3 \simeq \mathbb{Z}_3 \times Q_8$	
		$ S $	Description	$ S $	Description	$ S $	Description
$-\frac{1}{3}$	32	4	\mathbb{Z}_3 -orbifold	1 1	\mathbb{Z}_3 -orbifold conifold	1 1	\mathbb{Z}_3 -orbifold conifold
$-\frac{1}{4}$	24	1	conifold	1	conifold	1	conifold
$-\frac{1}{8}$	3	1	Q_8 -orbifold	1	\mathbb{Z}_8 -orbifold	1	Q_8 -orbifold
$-\frac{1}{12}$	24	1	conifold	1	conifold	1	conifold
$-\frac{1}{18}$	0 smooth, $S = \emptyset$					
$\frac{1}{24}$	1	1	G_1 -orbifold	1	G_2 -orbifold	1	G_3 -orbifold
∞	12	1	\mathbb{Z}_2 -orbifold	1	\mathbb{Z}_2 -orbifold	1	\mathbb{Z}_2 -orbifold
0	∞ large complex structure limit, $ S = \infty$					

Table 3: Singularities in the complex structure moduli space of the one-parameter family $X_\varphi^{(i)} = \tilde{X}_\varphi/G_i$, $i \in \{1, 2, 3\}$. For example, consider the singular variety $X_{-1/3}^{(3)}$. Its covering space has 32 isolated singular points, each contributing 1 to the total Milnor number $\tilde{\mu} = 32$. Under the G_3 -action, they form one orbit of length 8 and one orbit of length 24. Therefore, $X_{-1/3}^{(3)}$ has two singular points, one is a \mathbb{Z}_3 quotient of a conifold and the other is the usual conifold.

- $\mu_\varphi = \mu(P_\varphi)$ is the equivalent number of conifolds (that is, Milnor number) on the quotient $\tilde{X} = X/G$. Since we can identify, at each fixed point orbit $G \cdot p$, a normal subgroup $S_p \subset G$ as the stabilizer, the Milnor number on the quotient and Milnor number $\tilde{\mu}$ on the covering space are related as

$$|S_p| \tilde{\mu}(p) = \mu(G \cdot p).$$

- u_φ is the total vanishing cycle of the conifolds, that is, the sum of the vanishing cycles.
- σ is the symplectic form in the Frobenius basis (not: the basis where the symplectic form is the standard Σ matrix eq. (2.4)). As before, we have

$$\sigma = \frac{1}{Y_{111}} \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{pmatrix}$$

though for this manifold we have $Y_{111} = 4$ so the the anti-diagonal entries of σ are different from those for the $(4, 1)$ -manifold in eq. (2.4).

φ	$\tilde{\mu}(P_\varphi)$	μ	u_φ	U_φ
$-\frac{1}{3}$	32	12 (or 4)	$(8, 3, \frac{7}{6}, \frac{3}{4})$	$(1, -7, 8, 3)$
$-\frac{1}{4}$	24	1	$(24, 11, 5, \frac{13}{4})$	$(4, -28, 24, 11)$
$-\frac{1}{8}$	3	8	$(3, 2, \frac{5}{4}, 1)$	$(1, -6, 3, 2)$
$-\frac{1}{12}$	24	1	$(0, 1, 1, \frac{5}{4})$	$(1, -4, 0, 1)$
$\frac{1}{24}$	1	24	$(1, 0, \frac{1}{12}, 0)$	$(0, 0, 1, 0)$
∞	12	2	$(12, 3, 1, \frac{3}{4})$	$(1, -6, 12, 3)$

Table 4: Vanishing cycles at the six conifold points. The coordinates u_φ are relative to the ϖ -basis, and U_φ is relative to the integral symplectic Π -basis.

There is a slight complication at $\varphi = -\frac{1}{3}$ where it turns out that there are always multiple G -orbits that, moreover, arrange into orbits of different size under G_1 vs. G_2 and G_3 . In the following, we simply pick the larger $\mu_{-1/3} = 12$ leading to the shorter vector $u_{-1/3}$. With this convention, all relevant data for the vanishing cycles is listed in Table 4.

Let us return now to the Riemann symbol and make the change of variable

$$\varphi = \frac{\tilde{\varphi}}{1 - 18\tilde{\varphi}}, \quad \text{so} \quad \tilde{\varphi} = \frac{\varphi}{1 + 18\varphi}.$$

Using the properties of the Riemann symbol we see that (3.4) is equivalent to the scheme

$$\frac{1}{1 + 18\varphi} \mathcal{P} \left\{ \begin{array}{cccccccc} 0 & \infty & \frac{1}{42} & \frac{1}{18} & \frac{1}{15} & \frac{1}{14} & \frac{1}{10} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 2 & 2 & 2 & 2 & 2 & 2 \end{array} \quad \frac{\varphi}{1 + 18\varphi} \right\}. \quad (3.6)$$

The change of variables has interchanged the roles of the singularities at $\varphi = \infty$ and $\varphi = \frac{1}{18}$ with those at $\tilde{\varphi} = \frac{1}{18}$ and $\tilde{\varphi} = \infty$. The six conifold points now appear on an equal footing in the symbol. Based on this new symbol we can choose new period vector

$$\tilde{\varpi}(\tilde{\varphi}) = (1 + 18\varphi) \varpi(\varphi)$$

This satisfies a slightly simpler Picard-Fuchs equation with coefficient functions that are sixth order polynomials rather than seventh order, for detail see [18, §4.3].

3.3. Prepotential

Let us quickly review the general structure of the prepotential. For simplicity, we take the Kähler cone to be simplicial and spanned by $(1, 1)$ -forms ω_i ,

$$\mathcal{K} = \text{span}_{\mathbb{R}_{\geq}} \left\{ e_1, \dots, e_{h^{11}} \right\}.$$

Then, in the special coordinates $(z^0, \dots, z^{h^{11}})$, the prepotential takes the form

$$\mathcal{F}(z^0, \dots, z^{h^{11}}) = \mathcal{F}^{\text{pert}} + \mathcal{F}^{\text{np}}$$

whose perturbative part equals

$$\begin{aligned} \mathcal{F}^{\text{pert}} &= -\frac{1}{3!} \sum_{a,b,c=0}^{h^{11}} \frac{Y_{abc} z^a z^b z^c}{z^0} \\ &= -\frac{1}{3!} \sum_{i,j,k=1}^{h^{11}} \frac{z^i z^j z^k}{z^0} \int_X e_i e_j e_k - \frac{1}{4} \sum_{j,k=1}^{h^{11}} z^j z^k \int_X i^* i_!(e_j) e_k \\ &\quad + \frac{1}{24} \sum_{k=1}^{h^{11}} z^0 z^k \int_X c_2(X) e_k + \frac{1}{2} (z^0)^2 \chi(X) \frac{\zeta(3)}{(2\pi i)^3}, \end{aligned}$$

where $i : X \hookrightarrow \mathbb{P}_{\nabla}$ is the embedding map into the ambient toric variety.

Given an embedding as a hypersurface, the natural $(1, 1)$ -class is the pull-back of the canonical class of the ambient space, $i^*(K_{\nabla})$. This is always an integral cohomology class, but might be further divisible in $H^2(X, \mathbb{Z})$. For example, the pull-back canonical class of the quintic is divisible by 5. If we furthermore consider free quotients of a toric hypersurface, we have both the embedding and the quotient map

$$i : \tilde{X} \hookrightarrow \mathbb{P}_{\nabla}, \quad q : \tilde{X} \longrightarrow X.$$

Now we have a further divisibility problem, as the pull-back of a primitive generator in $H^2(X, \mathbb{Z})$ need not be primitive. For example, the generator H^2 on the $\mathbb{Z}_5 \times \mathbb{Z}_5$ -quotient of the quintic pulls back (via q^*) to 5 times the generator on the quintic.

However, we note that this cannot happen in solely permutation quotients, since in this case the regular orbits under the group action are precisely the invariants. In the case at hand, there is a unique primitive Kähler class $\omega \in H^{1,1}(X, \mathbb{Z})$. Its pullback under the quotient map $q : \tilde{X} \rightarrow X$ is the primitive G -invariant class $\tilde{\omega} \in H^{1,1}(\tilde{X}, \mathbb{Z})$, which itself is the restriction of the Kähler class ω_{∇} of the ambient toric variety \mathbb{P}_{∇} . That is,

$$q^*(\omega) = \tilde{\omega} = i^*(\omega_{\nabla})$$

The numerical invariants on the covering space are, of course, simple computations in toric geometry. For the case at hand, including the rescaling due to the G -quotient, we compute

$$\begin{aligned}\int_X e^3 &= \frac{1}{24} \int_{\tilde{X}} \tilde{e}^3 = 4 \\ \int_X q! i^* i_!(q^* e) e &= \int_{\tilde{X}} i^* i_!(\tilde{e}) \tilde{e} = \frac{1}{24} \int_{\mathbb{P}_\nabla} e_\nabla^4 = 4 \\ \int_X c_2(X) e &= \frac{1}{24} \int_{\tilde{X}} c_2(\tilde{X}) \tilde{e} = 4 \\ \chi(X) &= \frac{1}{24} \chi(\tilde{X}) = 0.\end{aligned}$$

3.4. Integral Homology Basis

Using the numerical invariants above, we obtain

$$\mathcal{F}^{\text{pert}}(z^0, z^1) = -\frac{2}{3} \frac{(z^1)^3}{z^0} - (z^1)^2 + \frac{1}{6} z^0 z^1.$$

In terms of the coefficient φ of the polynomial, the special coordinates can be taken to be $(z^0, z^1) = (\varpi_0(\varphi), \varpi_1(\varphi))$. To relate the period vectors, we look at the asymptotic behavior around the large complex structure limit $t = 0$, where

$$t = \frac{z^1}{z^0} = \frac{1}{2\pi i} \log \varphi + \dots$$

In terms of the special coordinates, the period vector takes the form

$$\Pi(z^0, z^1) = \begin{pmatrix} \partial \mathcal{F} / \partial z^0 \\ \partial \mathcal{F} / \partial z^1 \\ z^0 \\ z^1 \end{pmatrix} \sim \begin{pmatrix} \frac{2}{3} t^3 + \frac{1}{6} t \\ -2t^2 - 2t + \frac{1}{6} \\ 1 \\ t \end{pmatrix}; \quad \varpi = \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix} \sim \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Hence, the basis transformation $\Pi = \rho \varpi$ between the two different bases for the periods must be

$$\rho = \begin{pmatrix} 0 & \frac{1}{6} & 0 & \frac{2}{3} \\ \frac{1}{6} & -2 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.7}$$

Expressed in the Π -basis, the monodromy matrices are $M_\varphi = \rho m_\varphi \rho^{-1}$. We find

$$\begin{aligned}
M_{-1/3} - \mathbb{1} &= \begin{pmatrix} -96 & -36 & 12 & -84 \\ 672 & 252 & -84 & 588 \\ -768 & -288 & 96 & -672 \\ -288 & -108 & 36 & -252 \end{pmatrix}, & M_{-1/4} - \mathbb{1} &= \begin{pmatrix} -96 & -44 & 16 & -112 \\ 672 & 308 & -112 & 784 \\ -576 & -264 & 96 & -672 \\ -264 & -121 & 44 & -308 \end{pmatrix}, \\
M_{-1/8} - \mathbb{1} &= \begin{pmatrix} -24 & -16 & 8 & -48 \\ 144 & 96 & -48 & 288 \\ -72 & -48 & 24 & -144 \\ -48 & -32 & 16 & -96 \end{pmatrix}, & M_{-1/12} - \mathbb{1} &= \begin{pmatrix} 0 & -1 & 1 & -4 \\ 0 & 4 & -4 & 16 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -4 \end{pmatrix}, \\
M_{-1/18} - \mathbb{1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & M_0 - \mathbb{1} &= \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
M_{1/24} - \mathbb{1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & M_\infty - \mathbb{1} &= \begin{pmatrix} -24 & -6 & 2 & -12 \\ 144 & 36 & -12 & 72 \\ -288 & -72 & 24 & -144 \\ -72 & -18 & 6 & -36 \end{pmatrix}.
\end{aligned}$$

Table 5: The integral monodromy matrices for the singularities of the $(1,1)$ -manifold.

As expected, the monodromy matrices in the Π -basis are integral, $M_\varphi \in \text{GL}(4, \mathbb{Z})$ and symplectic, $M_\varphi \Sigma M_\varphi^T = \Sigma$. Here, Σ is the standard symplectic matrix as in (2.4). We conclude that the M_φ are the monodromies in an integral symplectic basis for $H^3(X, \mathbb{Z})$. The values of φ where the manifold develops a conifold singularity still have the special form eq. (3.5) of the monodromy matrix. In the integral basis, this is

$$M_\varphi = \mathbb{1} - \mu_\varphi U_\varphi U_\varphi^T \Sigma^{-1}, \quad U_\varphi = \rho u_\varphi,$$

and we list them in Table 4. The vectors U_φ span the entire $H^3(X, \mathbb{Z})$ lattice. We note that, as in the three-generation manifold, the singularity closest⁴ to the origin also has a particularly simple vanishing cycle $U = (0, 0, 1, 0)$ in the integral basis.

3.5. Instanton Numbers

The properly normalized Yukawa coupling is

$$\begin{aligned}
y_{\varphi\varphi\varphi} &= \frac{1}{\varphi^3} \exp \left(-\frac{1}{2} \int d\varphi \frac{R_3(\varphi)}{R_4(\varphi)} \right) \\
&= -384 \frac{\varphi + \frac{1}{18}}{\varphi^3 \left(\varphi + \frac{1}{3} \right) \left(\varphi + \frac{1}{4} \right) \left(\varphi + \frac{1}{8} \right) \left(\varphi + \frac{1}{12} \right) \left(\varphi - \frac{1}{24} \right)}
\end{aligned}$$

where we have fixed the integration constant to match the classical intersection number (times φ^{-3}) at the large complex structure limit $\varphi = 0$. The mirror Yukawa coupling is

$$y_{ttt} = \frac{1}{\varpi_0^2} y_{\varphi\varphi\varphi} \left(\frac{1}{2\pi i} \frac{d\varphi}{dt} \right)^3.$$

⁴That is, the singularity that determines the radius of convergence of the power series for the fundamental period.

We have to invert the mirror map eq. (3.7) to express φ in terms of t or, rather, $q \stackrel{\text{def}}{=} \exp(2\pi it)$. The mirror map and its inverse are

$$\begin{aligned} q(\varphi) &= \varphi + 4\varphi^2 + 48\varphi^3 + 680\varphi^4 + 10084\varphi^5 + 173056\varphi^6 + 3011300\varphi^7 + \mathcal{O}(\varphi^8) \\ \varphi(q) &= q - 4q^2 - 16q^3 - 40q^4 + 604q^5 - 9872q^6 + 56092q^7 + \mathcal{O}(q^8). \end{aligned}$$

We therefore obtain the instanton-corrected Yukawa coupling

$$\begin{aligned} y_{ttt} &= 4 + 12q - 116q^2 + 6924q^3 - 64884q^4 + 2146012q^5 - 22118516q^6 + \mathcal{O}(q^7) \\ &= 4 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1 - q^k} \end{aligned}$$

where the instanton numbers n_k are recorded in Table 6. Up to an overall factor of 4, these have already appeared in [18] from a purely combinatorial point of view. The extra factor is of course there to due to the classical intersection number 4, and appears to be only meaningful in the geometric setting.

k	n_k	k	n_k	k	n_k
1	12	7	1768032	13	5252822116016
2	-16	8	-12810048	14	-47718467477584
3	256	9	226260008	15	890108488876160
4	-1012	10	-1831410544	16	-8340130846927456
5	17168	11	33000429000	17	158096635640838140
6	-102432	12	-286340050052	18	-1512328959263997360

Table 6: Instanton numbers of the $(1,1)$ -threefold.

3.6. Cross Ratios

The cross ratio

$$X(z_1, z_1; z_2, z_3) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is invariant under reparametrizations of the \mathbb{P}^1 . Some of the cross-ratios of the complex structure singularities for the $(1,1)$ and $(4,1)$ manifold are the same:

$$X\left(0, -\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}\right) = X\left(0, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{12}\right)$$

But not all cross ratios agree, proving that the complex structure moduli spaces are different.

One might wonder whether there is any remaining symmetry of the 1-dimensional complex structure parameter spaces. However, one can check that there is no permutation of the 8 singularities preserving the cross ratios. In other words, there is no non-trivial subgroup of $\text{PSL}(2, \mathbb{C})$ mapping the singularities to themselves. Hence, the complex structure moduli space has no holomorphic⁵ symmetries.

⁵The only symmetry is complex conjugation.

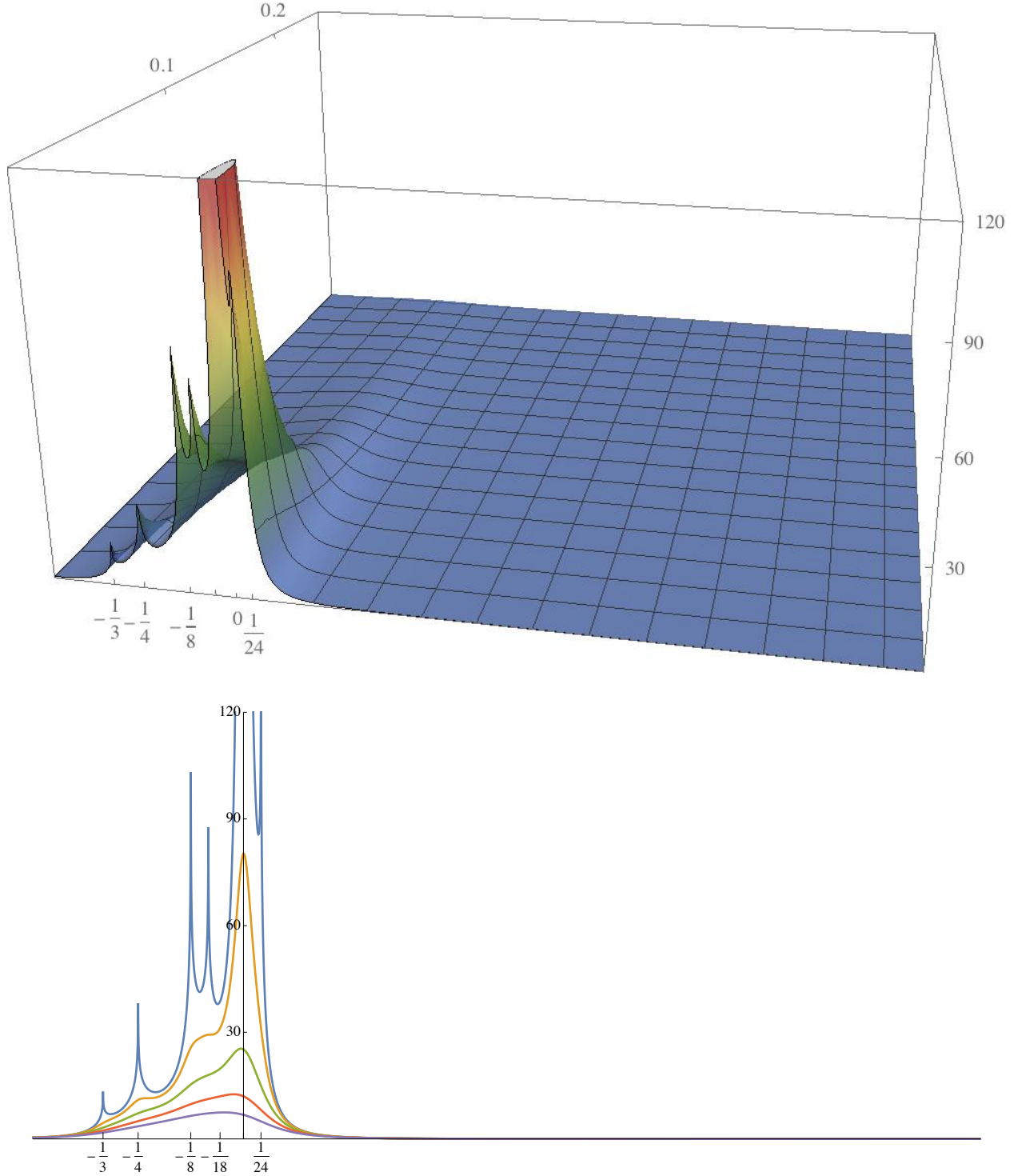


Figure 3: The metric $g_{\varphi\bar{\varphi}}$ on the complex structure moduli space. Top: plot over a region of the complex plane. Bottom: restriction to a real interval and its shift to constant imaginary values. These figures are included for comparison with Figure 2 and are plotted to cover the same range of φ .

4. Lattice Walks

4.1. Spectral Considerations

The countably infinite vertices of the lattice can be enumerated, say, $\{v_i\}_{i \in \mathbb{Z}}$. The allowed steps of the walk then define a matrix

$$A_{ij} = \begin{cases} 1 & \text{if we can step from } v_i \text{ to } v_j, \\ 0 & \text{else.} \end{cases}$$

This infinite symmetric matrix is the adjacency matrix of the graph whose nodes are the vertices and the edges are the admissible steps. By it being symmetric we can apply the spectral theorem and write

$$A = \int \lambda \cdot \rho_\lambda \rho_\lambda^T d\lambda,$$

where the integral ranges over all eigenvalues with associated (orthonormal) eigenvector density ρ_λ . Of particular interest will be the diagonal entries. Since there is no preferred vertex, the measure part must all be the same and equal to the eigenvalue density $\mu(\lambda)$:

$$|\rho_{\lambda,i}|^2 d\lambda = \mu(\lambda) d\lambda \quad \forall i.$$

Due to the combinatorics of the matrix product, the powers $(A^n)_{ij}$ count the number of lattice walks from v_i to v_j . Hence, the generating function for the number of lattice walks of length n starting and ending at the origin is

$$\varpi_0(\varphi) = \sum_n (A^n)_{0,0} \varphi^n = \sum_n \int (\varphi \lambda)^n \mu(\lambda) d\lambda = \int \frac{\mu(\lambda) d\lambda}{1 - \varphi \lambda}$$

We note that the power series must diverge outside of a finite disc around the origin, but right hand side need not. Of course this is nothing but the Borel resummation of the series.

The lattice Laplacian is the closely related matrix $\Delta = D - A$ where D is the *degree* matrix, that is, the diagonal matrix whose entry D_{ii} equals the number of adjacent nodes of the i -th node. In particular, in all cases that we will consider, the degree matrix is the identity matrix times the number of nearest neighbors a of the lattice. As in the continuous case, the spectrum of the Laplacian is real and non-negative. It is also bounded above by the lattice cutoff Λ , hence a subset of a finite real interval. Therefore, the spectrum of the adjacency matrix is also contained in a finite real interval

$$\text{Spec } \Delta = [0, \Lambda] \quad \Leftrightarrow \quad \text{Spec } A = [a - \Lambda, a]$$

The integrand of eq. (4.1) is only well-defined away from the reciprocal eigenvalues, that is, on $(-\infty, \frac{1}{a-\Lambda}] \cup [\frac{1}{\Lambda}, \infty)$. It is also bounded by $1/(1 - \varphi \hat{\lambda})$, where $\hat{\lambda} = \max(|a - \Lambda|, a)$ is the largest absolute value eigenvalue of A . Hence it converges on the disc $|\varphi| \leq 1/\hat{\lambda}$, which therefore is the disk where the power series converges. The power series expression for $\varpi_0(\varphi)$ necessarily diverges outside of that disk, but eq. (4.1) need not. In fact, away from the real line the denominator is uniformly bounded

$$|1 - \varphi \lambda| = \sqrt{(1 - \lambda \text{re}(\varphi))^2 + (\lambda \text{im}(\varphi))^2} \geq \begin{cases} \frac{1}{4}; & \text{if } |\lambda \text{re}(\varphi)| \leq \frac{1}{2} \\ \frac{1}{2} \left| \frac{\text{im}(\varphi)}{\text{re}(\varphi)} \right|; & \text{else,} \end{cases}$$

hence eq. (4.1) converges everywhere away from the real line. We conclude that the singularities of the generating function (or fundamental period) $\varpi_0(\varphi)$ are restricted to the real line, and start at distance $1/\hat{\lambda}$ away from the origin.

4.2. Square Lattice

As an example, consider the square/cubical/hyper-cubical d -dimensional lattice \mathbb{Z}^d and allow steps to the $2d$ adjacent vertices. Equivalently, consider the Calabi-Yau hypersurface in $(\mathbb{P}^1)^d$. Let $c_n^{(d)}$ be the number of lattice walks starting and ending at the origin. The sum of coordinates mod 2 always changes at each step, so there can only be walks of even length: $c_{2n+1}^{(d)} = 0$. In low dimensions, explicit formula are known to be

$$c_{2n}^{(1)} = \binom{2n}{n}, \quad c_{2n}^{(2)} = \binom{2n}{n}^2$$

though the apparent pattern does not continue with the cubic (3d) lattice. The generating function of the square lattice can be expressed in terms of the complete elliptic integral of the first kind⁶ as

$$\varpi_0^{(2)}(\varphi) = \sum c_{2n}^{(2)} \varphi^{2n} = \sum \binom{2n}{n}^2 \varphi^{2n} = \frac{2}{\pi} K(4\varphi),$$

which is singular at $\varphi = \pm \frac{1}{4}$. We will follow the conventional choice of branch cuts $(-\infty, -\frac{1}{4}] \cup [\frac{1}{4}, \infty)$. The power series of course diverges outside of the disc $|\varphi| \leq \frac{1}{4}$.

There are various finite-dimensional approximations to the infinite lattice, for the purposes of this example we consider it on a torus with $n \geq 3$ lattice points⁷ in each direction, for a total of n^d lattice points. For $d = 1$, we get the circle graph whose adjacency matrix has eigenvalues

$$A^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \Rightarrow \lambda_i^{(1)} = 2 - 4 \sin\left(\frac{\pi i}{n}\right)^2, \quad i = 0, \dots, n-1.$$

The eigenvalues for the Cartesian product graph are the sums of the factor eigenvalues, that is,

$$\lambda_{(i_1, \dots, i_n)}^{(d)} = 2d - 4 \sum_{j=0}^d \sin\left(\frac{\pi i_j}{n}\right)^2, \quad (i_1, \dots, i_d) \in \{0, \dots, n-1\}^d.$$

The spectral version of the square lattice generating function is, therefore,

$$\varpi_0^{(2)}(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} \frac{1}{1 - 4\varphi \left(1 - \sin\left(\frac{\pi i}{n}\right)^2 - \sin\left(\frac{\pi j}{n}\right)^2\right)}$$

By truncating the power series and spectral series presentations, we can compute approximations. In Figure 4, we plot all three and visualize their convergence.

⁶Note that $K(x) = \text{elliptic_kc}(x^2)$ in Sage.

⁷We assume that $n \geq 3$ to ensure that the number of adjacent vertices is still $2d$.

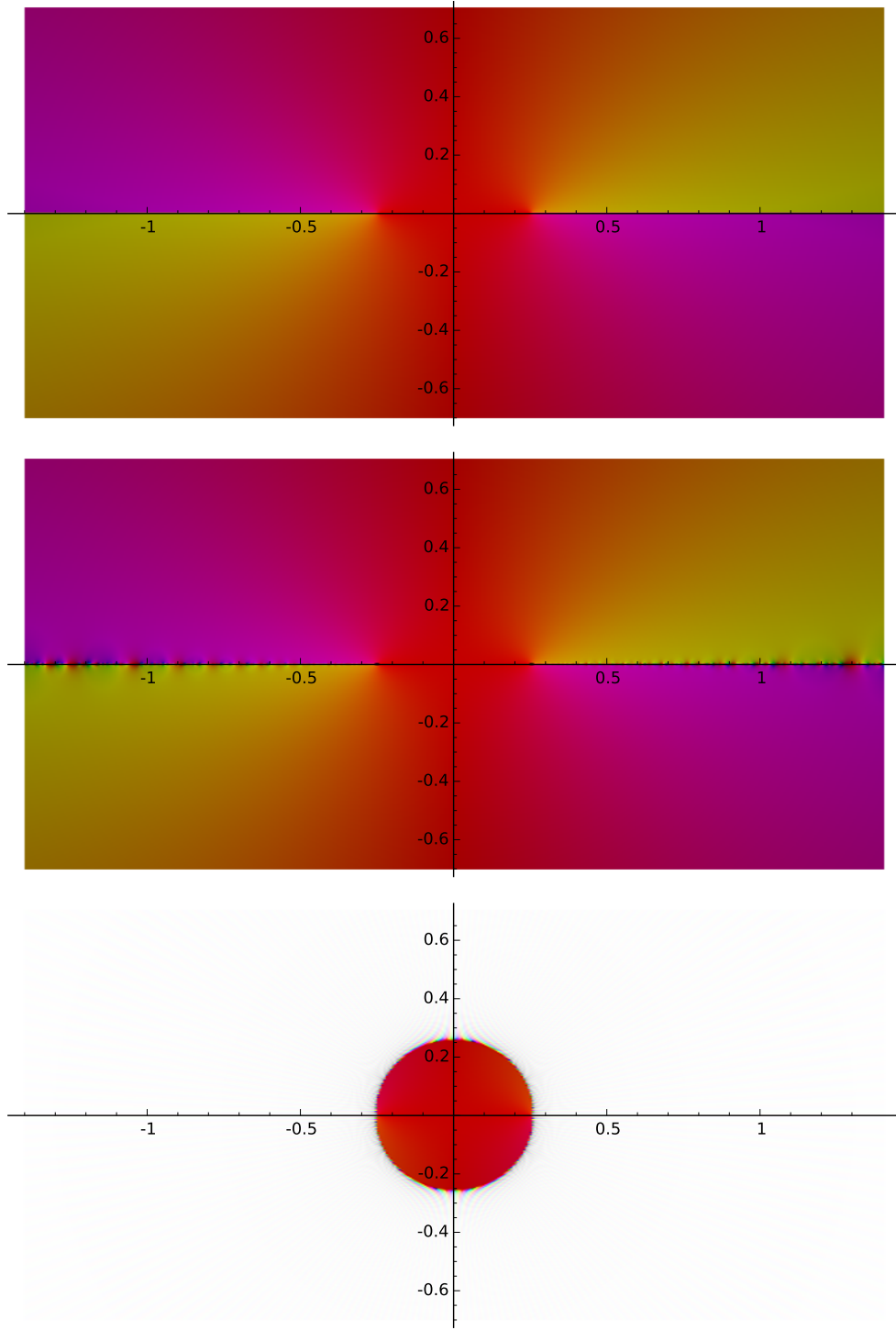


Figure 4: The square lattice generating function. Top: The exact solution using the complete elliptic integral. Middle: The spectral sum eq. (4.1) with $n = 75$. Bottom: The truncated series $\sum_0^{99} c_{2n}^{(2)} \varphi^{2n}$. Brightness is magnitude and hue is phase.

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Note Added

After this paper was completed we became aware of the paper [19] by Kawada *et al.* which has a substantial overlap with the present work.

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